

THE MINISTRY OF EDUCATION AND SCIENCE OF THE RUSSIAN FEDERATION
Federal State Autonomous Educational Institution of Higher Education
Lobachevsky State University of Nizhni Novgorod
National Research University

M.V. Kemaeva

ECONOMIC MATHEMATICAL MODELS

TUTORIAL

Nizhni Novgorod

2017

МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ РОССИЙСКОЙ ФЕДЕРАЦИИ

**Федеральное государственное автономное образовательное учреждение
высшего образования**

**«Национальный исследовательский Нижегородский
государственный университет им. Н.И. Лобачевского»**

М.В. Кемаева

МАТЕМАТИЧЕСКИЕ МОДЕЛИ В ЭКОНОМИКЕ

Учебно-методическое пособие
по дисциплине «МАТЕМАТИЧЕСКИЕ МОДЕЛИ В ЭКОНОМИКЕ»

Рекомендовано методической комиссией Института экономики и
предпринимательства ННГУ для иностранных студентов, обучающихся по
направлению подготовки 38.03.01 «Экономика» (бакалавриат) на английском
языке

Нижний Новгород

2017

УДК 330
ББК 65.05
К-35

К-35 **М.В. Кемаева.** МАТЕМАТИЧЕСКИЕ МОДЕЛИ В ЭКОНОМИКЕ: Учебно-методическое пособие. – Нижний Новгород: Нижегородский госуниверситет, 2017. – 46 с.

Рецензент: к.э.н., доцент Ю.А.Гриневич

В настоящем пособии изложены учебно-методические материалы по курсу «МАТЕМАТИЧЕСКИЕ МОДЕЛИ В ЭКОНОМИКЕ» для иностранных студентов, обучающихся в ННГУ по направлению подготовки 38.03.01 «Экономика» (бакалавриат).

Пособие дает возможность бакалаврам расширить основные знания о методах экономического анализа, которые объединяют экономическую теорию со статистическими и математическими методами анализа, овладевать умением комплексно подходить к вопросам экономического развития, использовать различные источники информации; развивать экономическое мышление.

Ответственный за выпуск:
председатель методической комиссии ИЭП ННГУ,
к.э.н., доцент Летягина Е.Н.

УДК 330
ББК 65.05

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Introduction.

The term "model" is widely used in various spheres of human activity and has many meanings. The word "model" has its origin from the Latin 'modulus', which means measure, norm or sample. We will restrict ourselves to the following understanding of the word model: under the "model" is a visualize or materially implemented system, which replaces the real object (the system) maintaining some of its most important features in the process of learning, analysis. Its analyses can gives us new information about the object.

Thus, the model can be defined as a conditional image (simplified) of a real object (process) that is created for a more depth study of reality.

Research methods, based on the development and usage of models, called modeling. Modeling in scientific research was used even in ancient times and gradually captured a whole new field of scientific knowledge. Currently, simulation methods are used in almost all fields of research.

Thus, the modeling refers to the process of building, study and application of models. It is closely related with categories such as abstraction, analogy, hypothesis, etc. The main feature of the simulation that it is the method of indirect cognition of objects. The model acts as a tool of knowledge that the researcher puts between himself and the object and examines using different tools.

The need for modeling is formed by the difficulty and sometimes impossibility of direct study of a real object (process). It is easier to create and study the prototypes of real objects, i.e. model. We can say that the theoretical knowledge about something, usually consists of various models. These models reflect the intrinsic properties of a real object (process), although in fact the reality is much richer and richer. The goal of modeling is to understand and study the qualitative and quantitative nature of the phenomenon, to reflect significant study the features of the phenomena suitable to use in practical activities. The modeling methodology can be summarized in the following main stages:

1. Model building

The model always assumes some knowledge about the object (the original). The cognitive capabilities of the model is formed by reflecting the most essential features of the object-of the original. The study of one of the sides of the object being modeled is the price of failure from the reflection of the other parties. Therefore, any model replaces the original only in a strictly limited sense. It follows that for the same object can be built several "specialized" models focused

on specific sides of the examined object or characterize an object with varying degrees of detail.

2. The research model.

Once the model is built, it acts as an independent object of study. One form of such research is to conduct model experiments. It is needed to change consciously functioning of the model and systematized data about its "behavior". The results are a lot of knowledge about the model.

3. The forming of knowledge about the object of study.

The third stage is the transfer of knowledge from model to the original, the formation of additional knowledge about the object. This process of knowledge transfer is carried out according to certain rules. Knowledge about the model must be adjusted to reflect those properties of the original, which is not reflected or were changed in the model.

4. Model validation and practical use of the obtained knowledge.

At this stage the practical verification obtained using models of knowledge and its use to build a General theory of the object, its transformation or control. To understand the essence of modeling it is important to keep in mind that modeling is not the only source of knowledge about the object. The modeling process "immersed" in a more General process of cognition. This circumstance is taken into account not only at the stage of model building, but at the final stage, when the unification and generalization of results obtained should be made on the basis of diverse funds of knowledge.

Modeling is the cyclic process. It means that for the first four-cycle may be followed by second, third, etc. When this knowledge about the studied object is expanded and refined, the original model is to be gradually improved. The defects detected after the first cycle of the simulation can be corrected in subsequent cycles, due to the small knowledge of the subject and the errors in the model.

Modeling is often compared with an alternative method of exploring reality: the scientific method and experimentation. Advantages of the method of modeling are: versatility, lower cost, shorter duration of time, etc.

The shortcomings of the modeling method are: the necessity of collecting a large amount of reliable information; the epistemological difficulties of constructing an adequate model, because you can never be sure of the adequacy of the model; the object of modeling may be changed (the model successfully worked in the past may not necessarily be useful at the present); the boundaries of applicability of the model is generally unknown (the results of some simulation experiments can be useful, others not); the research and development of a model can be much more expensive than anticipated.

We should dwell on the concept of adequacy of the model.

The model, which successfully achieved the goal, would be adequate for this circuit. Adequacy means that the requirements of completeness, accuracy and correctness (truth) models are not General but the extent that is sufficient to achieve the goal.

The adequacy of the model should not be confused with proximity. The closeness of the model may be very high, but in all cases, the model is another object and the inevitable differences (the only perfect model of any object is the object itself). The amount, measure, degree of acceptance of differences, you can define only in correlation with the purpose of modeling.

Chapter 1. Intersectoral Balance.

1.1. Intersectoral Balance: introduction.

Economic planning needs to determine the volume of production of goods, providing the specified demand and production needs at the level of regions, country or any other object including several elements. It is possible to solve this task using the balance models of production and distribution. The basis of the construction of these models lies in the balance method, i.e. a method of mutual matching of the available material, human and financial resources with the need for them.

Balance planning methods can be considered at different levels of the economic hierarchy: companies, associations, industries, national economy. The model input-output balance is historically the first economic-mathematical model of the summary economic planning. The first balances of the national economy were developed by the Central statistical administration of the USSR in 1923-1924. Currently interindustry balances at the national level forms approximately eighty countries around the world. Also, there are intersectoral balances at the level of regions and cities.

Predecessors intersectoral balances were: economic table Quesnay F. (1758) and schemes of social reproduction Marx K. (XIX century). Russian economist V. Dmitriev (1868-1913), studying cross-sectoral links, he was the first who used linear equations and offered technological factors for this purpose. The author of modern input-output models is an American scientist (Russian origin) Wassily Leontief. In 1973, his methods of economic analysis (model “input-output”), he was awarded the Nobel prize.

This model allows to calculate the total cost gross production, direct and indirect costs per unit of output, and also gives the opportunity to establish clear quantitative correlation between gross social product, national income, development of specific industries.

The method is universal method. For example, intersectoral balances were used by Americans to carry out a restructuring of the economy after Second World War. It was taken in the basis of indicative plans used in Japan.

Interbranch balance of production and distribution - a tool of analysis and planning the structure of social production, taking into account the complex links

formed in production sectors. Depending on measurements of the units flows of analyzing in the balance, there are different types of inter-sectoral balance: monetary, naturally-cost, labor gauges. Balances can be divided on the economic content of information into planning and reporting; on the nature of the used model - static and dynamic.

Lets consider the fragment (three sections) reporting input-output balance, in which the flows of products are measured on the basis of the value of production in some fixed price (table 1). The balance is the set of units of material production. Table 1 is only table shown main links between elements of analyzing the system but not yet a mathematical model.

Table 1. A fragment of the input-output table

		Intermediate good					Final product	Gross product
		1	2	...	n	Итого		
current material costs	1	a_{11}	a_{12}	...	a_{1n}	$\sum_{j=1}^n a_{1j}$	$\dots Y_1$	X_1
	2	a_{21}	a_{22}	...	a_{2n}	$\sum_{j=1}^n a_{2j}$	$\dots Y_2$	X_2

	n	a_{n1}	a_{n2}	...	a_{nn}	$\sum_{j=1}^n a_{nj}$	Y_n	X_n
	Total	$\sum_{i=1}^n a_{i1}$	$\sum_{i=1}^n a_{i2}$...	$\sum_{i=1}^n a_{in}$	$\sum_{i=1}^n \sum_{j=1}^n a_{ij}$	$\sum_{i=1}^n Y_i$	$\sum_{i=1}^n X_i$
Added value		V_1	V_2	...	V_n	$\sum_{j=1}^n V_j$	IV	
Gross product		X_1	X_2	...	X_n	$\sum_{j=1}^n X_j$		

Each branch appears twice in the balance: as producing and as consuming. Industry as the manufacturer of the products corresponds to a particular i-row of the table, and as consumer products - a particular j-column. As industries are clean, the index of the industry can be identified as a product so as technological process.

The first section of intersectoral balance provides information about the linkages. The value a_{ij} located at the intersection of the branches (i.e. rows and columns) should be understood as the value of the means of production, produced

in the i-industry and consumed as material costs in j- industry (inter-industry product supply due to industrial activity sectors). .

Thus, each i-line of the first section shows the distribution of the j-branch between other sectors of the economy. $P_i = \sum_{j=1}^n a_{ij}$ - production and consumption of products in the i-branch of the economic system (the intermediate product the i-branch).

In the columns of the first section of the balance sheet reflects the structure of the material costs of each industry. $Z_j = \sum_{i=1}^n a_{ij}$ - total production costs j-industry in the reporting period. $Z = \sum_{j=1}^n \sum_{i=1}^n a_{ij}$ - total production costs of all branchess or total intermediate product of the national economy.

Thus, the first section of intersectoral balance shows the overall picture of production costs and product distribution industries for production purposes. The data of the first quadrant play a crucial role in the analysis of the structure of the material cost of industries, proportions and input-output linkages between industries, streams in system of logistics.

The second section provides value Y_i - value of the final product, and X_i - the value of gross domestic product ($i = \overline{1, n}$).

The final product is the product of material production sectors, arriving at the goal of personal and public non-production consumption, accumulation and compensation disposal of fixed assets, inventories, expenditures on education, health, export etc.

$\sum_{i=1}^n Y_i$ - total final product of the economic system or national income, and the column Y characterizes the material structure of national income.

The final product of each industry could be shown differentially, according to the directions of use in more advanced schemes balance: for consumption, investment, inventories and reserves, exports, and other expenses.

The first and second sections of the input-output balance is called the table of "input-output". The rows of this table is built with the following carrying value:

$$X_i = \sum_{j=1}^n a_{ij} + Y_i, \quad (i = \overline{1, n}), \quad (1.1)$$

i.e. the gross product of each industry is equal to the sum of final and intermediate products.

The third section of intersectoral balance reflects the cost structure of gross product by industries. In table 1 the third section presents in 2 rows. Values V_j in the first mean value added by industries. X_j in the second - gross domestic product. V_j are defined as the difference between gross production and total production costs:

$$V_j = X_j - \sum_{i=1}^n a_{ij}, \quad j = \overline{1, n} \quad (1.2)$$

Value added is the portion of the cost of the product, which is created in this industry. It includes profits, wages, amortization, taxes and other costs incurred by each object (industry) in addition to payments for resources from other analyzing elements.

Values added of deployed intersectoral balance are usually divided into amortization and net products.

From equations (1.1) and (1.2) should

$$\sum_{j=1}^n \left(V_j + \sum_{i=1}^n a_{ij} \right) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} + Y_i \right) \quad (1.3),$$

$$\text{Whence we obtain: } \sum_{j=1}^n V_j = \sum_{i=1}^n Y_i \quad (1.4)$$

This ratio shows that total final product of the economic system (national income) is equal to the total value added. Thus, the third section also describes the national income, but in its cost structure as the sum of wages and net income of all branches of material production, the values V_j show the industry's contribution to national income.

The data from the third section is required for analysis of the relationship between the newly created and transferred value, between necessary and surplus product in the whole material production and by the branches. In General, equation (1.4) shows that in the interindustry balance is realized the most important principle of the unity of material and cost composition of national income.

It should be noted that the balance of natural values usually contains only the sectors I and II of the table 1 (the input-output balance). It is developed on the most important products and usually does not cover whole production.

1.2. Static Balance Model of the Production.

Carrying out the Static Balance Model is based on the following assumptions about the economic properties of the object:

- Economic system consists of several economic entities. The quantity of each object's product can be characterized by one number, which is often seen as gross output in some fixed price.
- Products of each object are consumed partially by other objects of the system and come out partially as the final product of this system, i.e., it is correct that

$$X_i = \sum_{j=1}^n a_{ij} + Y_i \quad i = \overline{1, n}; \quad (1.5)$$

- The goal of the system is to produce a given quantity of the final product.
- The property of completeness of consumption: to produce a specified quantity of a product object must obtain a certain number of other products.
- The property of linearity of consumption: an increase in output in a number of times requires an increase in the consumption of all other products in the same number of times.

Obviously, formulated assumptions are only approximately reflect the real economic situation. For example, the assumption of completeness of consumption, which implies that the production technology in each object remains unchanged during the time interval under consideration, and in each sector there is a single production technology, that are not allowed substitution of one input for another.

In the actual production of the same product used may require different amounts of ingredients depending on the technology, and the model assumes that the product is described in some average way. Despite these simplifications, the balance model is a useful planning tool because of its simplicity and the possibility of calculation all indicators of the plan.

1.2.1. The construction of the model.

Lets select variables of a model the gross output (X_i , ($i = \overline{1, n}$)). Due to the assumptions 2, part of these products leaves the system as the final product Y_i . The values Y_i are considered in the model as a planed task and the relation (1.5) is correct:

$$X_i = \sum_{j=1}^n a_{ij} + Y_i \quad (i = \overline{1, n};)$$

The properties of linearity and completeness of consumption define the patterns of transformation of resources in the system. According to the property of completeness to produce a unit of output j-object must use other products of the economic system in a certain ratio. Let the vector $\bar{\alpha}_j = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{nj})$ that defines this relationship, and values α_{ij} are called technological coefficients or coefficients of direct costs.

α_{ij} - number of products i- industry needed to produce a unit of output in the j- industry. Values do not depend on production volume and are relatively stable values over time.

The matrix is composed of values α_{ij} is called the matrix of technological coefficients or matrix of direct costs

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix}$$

Following the economic sense of values α_{ij} , all elements of the matrix A is not negative. This property could be recorded in the following way: $A \geq 0$. As the process of production cannot be carried out, if industry has spent more products for its own production than was created, it is obvious that the diagonal elements of the matrix A is less than 1: $\alpha_{ii} < 1$

Based on the properties of linearity can be argued that: if j-object will produce X_j units of output, then it needs $\alpha_{ij} \cdot X_j$ units of i- branch, i.e. cross-sectoral delivery of i- branch in j- branch is equal to

$$a_{ij} = \alpha_{ij} \cdot X_j \quad (1.6)$$

We substitute (1.6) into (1.5) and obtain the following system of balance equations:

$$X_i = \sum_{j=1}^n \alpha_{ij} X_j + Y_i \quad (i = \overline{1, n};) \quad (1.7)$$

$$\text{From an economic sense value } X_j \geq 0 \quad (1.8)$$

The relations (1.7) and (1.8) together with the above interpretation of the coefficients α_{ij} of the vectors X u Y define a simple balance model Leontief.

In matrix form, the model can be written as follows:

$$\begin{cases} A \cdot X + Y = X \\ X \geq 0 \end{cases} \quad (1.9).$$

In carrying models are specified: the matrix A and the vector of the final product Y. The matrix X (gross output) should be determined.

When considering balance models, the question about the determination of the coefficients of the direct costs (the matrix A) is complicated. In the simplified model it is assumed that the coefficients of direct costs in the period under constant and depend only on the current production technology, and it allows us to calculate them on the basis of data on the actual production flow for the last period presented in the reporting intersectoral balance:

$$\alpha_{ij} = \frac{a_{ij}}{X_j} \quad (2.10)$$

1.2.2. The study of the system of balance equations .

Let's consider intersectoral balance model:

$$\begin{cases} A \cdot X + Y = X \\ X \geq 0 \end{cases} \quad (1.11)$$

$$(1.12)$$

Studying of the system of equations (1.11) means clarifying the conditions that guarantee the existence and uniqueness of nonnegative solutions of this system. Intersectoral balance model is a linear system of n-equations with n-variables. Such systems have a unique solution if the determinant is not zero. Let us introduce the unit matrix E, and we write (1.11) in the form:

$$(E - A) \cdot X = Y \quad (1.13)$$

Thus, the system of equations (1.11) had the solution if the determinant of the matrix (E-A) would be different from zero: ($\Delta(E - A) \neq 0$). In this case, there it exists a matrix $S = (E - A)^{-1}$ inverse to (E - A).

Then the solution of system (1.11) can be determined as follows:

$$X = (E - A)^{-1} \cdot Y \quad (1.14)$$

However, to ensure that the decision had an economic sense, its is needed that the decision is nonnegative, i.e $X \geq 0$. It should be noted that the existence of a matrix S does not ensure the nonnegative the solution is obtained. In addition, from an economic point of view, systems are practically interesting if they have a nonnegative solution for any targeted vector of the final product Y, i.e., for any positive Y ($Y > 0$).

Thus, the main question in the study of the Leontief model is the following: whether the technology, described by the matrix A , to provide any final demand $Y > 0$. From a mathematical point of view, it means identifying the conditions that must be met for the matrix A , so that there is any system of balance equations with non-negative solution. The answer to this question is related to the notion of productivity matrix A .

The matrix A is called productive if there exists a nonnegative vector $X > 0$ that

$$(E - A) \cdot X > 0, \text{ i.e. } X > A \cdot X \text{ (1.15).}$$

Condition (1.15) means that the quantity of product is more than goes for the industrial consumption (intermediate product $A \cdot X$). Therefore, each object produces a certain amount of the final product. In the case of the productive matrix A model (1.11-1.12) also called productive.

Theorem – 1. Productivity matrix A is a necessary and sufficient condition for the existence and uniqueness of nonnegative solutions of the system of balance equations (1.11).

Theorem - 2 (necessary and sufficient condition for productivity). The matrix A is productive if and only if there exists a matrix S and all its elements are non-negative.

Theorem - 3 (sufficient condition for productivity). The matrix A is productive if all its elements are nonnegative and the sum of the elements in each column is not greater than one ($\sum_{i=1}^n \alpha_{ij} \leq 1; j = \overline{1, n}$).

A sufficient condition can only be used for the matrix in monetary measures. In addition, it should be noted that the matrix can be productive and in case of failure to fulfill this condition (since this is a sufficient, not a necessary feature).

So, solution of the system of balance equations with productive matrix A can be written:

$$X = S \cdot Y \quad (1.16),$$

i.e. on the basis of specified coefficients of total costs and final product you can determine the gross production of industries. This is the main idea of intersectoral models for production planning. From the linearity of the Leontief model, it follows that the ΔY -increment vector Y and the corresponding ΔX - increment of the vector X are linked by the equation $\Delta X = S \cdot \Delta Y$. Therefore, the matrix allows us to calculate the change in gross output caused by the change in final consumption. Therefore, the matrix $S = (E - A)^{-1}$ is often called a matrix multiplier or multiplier Leontief.

1.2.3. The economic meaning of the matrix S.

We will denote s_{ik} $i = \overline{1, n}; k = \overline{1, n}$ the elements of the matrix S and determine its economic meaning. Consider a special case: let one unit of the final product produces by k- industry, and all other sectors do not produce final products, i.e.

$$Y_j = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} \quad (1.17)$$

If A is productive, $X = S \cdot Y$ i.e

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1k} & \dots & s_{1n} \\ s_{21} & s_{22} & \dots & s_{2k} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_{i1} & s_{i2} & \dots & s_{ik} & \dots & s_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_{n1} & s_{n2} & \dots & s_{nk} & \dots & s_{nn} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} s_{1k} \\ s_{2k} \\ \vdots \\ s_{ik} \\ \vdots \\ s_{nk} \end{pmatrix} \quad (1.18)$$

From the equality of the vectors in (1.18) it follows that $x_i = s_{ik}$ ($i = \overline{1, n}$) (1.19).

The relation (1.18) reveals the economic sense of the elements s_{ik} of the matrix S: s_{ik} is the gross amount of product, which should make i- branch to the k-branch to release one unit of the final product. Therefore, the elements s_{ik} are called the coefficients of the full material costs and matrix S - matrix of full material costs (material costs are the products manufactured by the objects of the consider economic system).

The coefficients of direct costs α_{ij} describe the direct costs of production i- industry in the production unit of j-branch. However, in addition to direct costs, there are indirect costs.

The matrix $K = S - E - A$ is called the matrix of indirect material cost and its elements k_{ij} shows indirect cost of per unit of production by j-industry. Indirect costs of higher order is very small, so in practice they can be neglected.

Example 1. Pulp and paper mill consists of 2 main divisions and one subsidiary that produce special type of product. For the production of paper each division must receive a certain amount of product from other economic facilities (divisions).

The products of each division is partially spent on internal consumption (intermediate product), and partially flows around the outside of the system as a final product.

The data presented in the table of input-output:

Division	Intermediate good			Final good y_t
	1	2	3	
1	10	5	40	45
2	30	0	30	40
3	20	40	0	140

1. To complete the balance.
2. To determine:
 - a. the coefficients of direct costs;
 - b. production program of divisions for the vector of the final product $Y=(200;100;300)$ and to build a mathematical model;
 - c. matrix full and indirect costs;
 - d. the change in gross output, when the change in final demand in the 1st division by 5%, and 3 by 40%.

Solution.

1. Let`s complete the balance.

Division	Intermediate good			Intermediate P	Final good Y	Gross prod. X
	1	2	3			
1	10	5	40	55	45	100
2	30	0	30	60	40	100
3	20	40	0	60	140	200
Итого:	60	45	70		225	400
V	40	55	130		225	
X	100	100	200			400

2. Let`s determine:
 - a. the matrix A (direct costs).

$$\alpha_{ij} = \frac{a_{ij}}{X_j}$$

α_{ij} -coefficients of direct costs or technological factors.

$$A = \begin{pmatrix} 0.1 & 0.05 & 0.2 \\ 0.3 & 0 & 0.15 \\ 0.2 & 0.4 & 0 \end{pmatrix}$$

b. production program of divisions for the vector of the final product $Y=(200;100;300)$ and to build a mathematical model;

A simple balance model Leontief:

$$\begin{cases} AX + Y = X \\ X > 0 \end{cases}$$

Will build a system of balance equations:

$$\begin{cases} 0.1X_1 + 0.05X_2 + 0.2X_3 + 200 = X_1 \\ 0.3X_1 + 0X_2 + 0.15X_3 + 100 = X_2 \\ 0.2X_1 + 0.4X_2 + 0X_3 + 300 = X_3 \end{cases} \quad (1)$$

$$\{X_i \geq 0 \quad (2)$$

Constraints (1) and (2) determine the economic-mathematical model of the problem.

Define the gross output of each division.

METHOD 1: Cramer's Rule.

$$x_j = \frac{\Delta_j}{\Delta},$$

Δ - the determinant of a matrix;

Δ_j is obtained from the determinant by replacing the coefficients in front of a variable X_j on the column free members.

After simplification of the system (1) we get:

$$\begin{cases} 0.9X_1 - 0.05X_2 - 0.2X_3 = 200 \\ -0.3X_1 + X_2 - 0.15X_3 = 100 \\ -0.2X_1 - 0.4X_2 + X_3 = 300 \end{cases} \quad (3)$$

$$\Delta \equiv \begin{vmatrix} 0.9 & -0.05 & -0.2 \\ -0.3 & 1 & -0.15 \\ -0.2 & -0.4 & 1 \end{vmatrix} = 0.7655$$

$$X_1 = \frac{\begin{vmatrix} 200 & -0.05 & -0.2 \\ 100 & 1 & -0.15 \\ 300 & -0.4 & 1 \end{vmatrix}}{0.7655} = \frac{263.25}{0.7655} = 343.89$$

$$x_2 = \frac{\begin{vmatrix} 0.9 & 200 & -0.2 \\ -0.3 & 100 & -0.15 \\ -0.2 & 300 & 1 \end{vmatrix}}{0.7655} = \frac{210.5}{0.7655} = 274.98$$

$$x_3 = \frac{\begin{vmatrix} 0.9 & -0.05 & 200 \\ -0.3 & 1 & 100 \\ -0.2 & -0.4 & 300 \end{vmatrix}}{0.7655} = \frac{366.5}{0.7655} = 478.77$$

Thus, the release plan for the 1st division - 343.89
 2nd division - 274.98
 3rd division - 478.77

METHOD 2: using the inverse matrix.

$$\begin{cases} AX + Y = X \\ X > 0 \end{cases} \Rightarrow X - AX = Y \rightarrow (E - A)X = Y$$

$$X = (E - A)^{-1}Y$$

$S = (E - A)^{-1}$ - matrix of the full costs of production.

$$1. E - A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.05 & 0.2 \\ 0.3 & 0 & 0.15 \\ 0.2 & 0.4 & 0 \end{pmatrix} = \begin{pmatrix} 0.9 & -0.05 & -0.2 \\ -0.3 & 1 & -0.15 \\ -0.2 & -0.4 & 1 \end{pmatrix}$$

$$2. \Delta = 0.7655$$

$$3. (E - A)^T = \begin{pmatrix} 0.9 & -0.3 & -0.2 \\ -0.05 & 1 & -0.4 \\ -0.2 & -0.15 & 1 \end{pmatrix}$$

$$4. A_{11} = (-1)^2(1 - 0.06) = 0.94$$

$$A_{12} = (-1)^3(-0.05 - 0.08) = 0.13$$

$$A_{13} = (-1)^4(0.0075 + 0.2) = 0.2075$$

$$A_{21} = (-1)^3(-0.3 - 0.03) = 0.33$$

$$A_{22} = (-1)^4(0.9 - 0.04) = 0.86$$

$$A_{23} = (-1)^5(-0.135 - 0.06) = 0.195$$

$$A_{31} = (-1)^4(0.12 + 0.2) = 0.32$$

$$A_{32} = (-1)^5(-0.36 - 0.01) = 0.37$$

$$A_{33} = (-1)^6(0.9 - 0.015) = 0.885$$

$$(E-A)^{-1} = \begin{pmatrix} \frac{0.94}{0.7655} & \frac{0.13}{0.7655} & \frac{0.2075}{0.7655} \\ \frac{0.33}{0.7655} & \frac{0.86}{0.7655} & \frac{0.195}{0.7655} \\ \frac{0.32}{0.7655} & \frac{0.37}{0.7655} & \frac{0.885}{0.7655} \end{pmatrix} = \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix}$$

$$X=(E-A)^{-1}Y$$

$$X = \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix} * \begin{pmatrix} 200 \\ 100 \\ 300 \end{pmatrix} = \begin{pmatrix} 343.9 \\ 275 \\ 478.7 \end{pmatrix}$$

c. matrix full and indirect costs

$$S = \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix} \text{ matrix of the full costs of production.}$$

$K=S-E-A$ – matrix of indirect costs

$$K = \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.05 & 0.2 \\ 0.3 & 0 & 0.15 \\ 0.2 & 0.4 & 0 \end{pmatrix} = \begin{pmatrix} 0.128 & 0.12 & 0.071 \\ 0.131 & 0.123 & 0.105 \\ 0.218 & 0.083 & 0.156 \end{pmatrix}$$

d. the change in gross output, when the change in final demand in the 1st division by 5%, and 3 by 40%.

$$\Delta X = S * \Delta Y$$

$$\Delta Y = \begin{pmatrix} 10 \\ 0 \\ 120 \end{pmatrix} \quad \Delta X = \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix} * \begin{pmatrix} 10 \\ 0 \\ 120 \end{pmatrix} = \begin{pmatrix} 44.8 \\ 34.91 \\ 142.9 \end{pmatrix}$$

1.3. Balance model with factors of production.

For the functioning of economic agents required not only the products of other objects of the system, but also such factors as production assets (equipment, production facilities, labor, etc.) Additionally, the economic system can obtain products from other economic systems. The volume of these factors are usually limited, it is the reason that not every vector of the final product can be produced

by the economic system, even if the matrix A is productive. Therefore, it is necessary to calculate the demand of system in the factors of production to determine a plan. A valid plan is the only plan which needs do not exceed the available quantities of factors.

The demand of system in the factors of production will be denoted by $Z = (z_1, z_2, \dots, z_m)$, where z_i is the need for i - factor. Demand can be measured in natural units (hours, square feet, etc, etc), and in monetary units. Each economic unit will be characterized by the vector of costs of factors of production per unit of output: $B_j = (\beta_{1j}, \beta_{2j}, \dots, \beta_{mj})$, here β_{ij} is the number of i -factor required for the j -object to produce a unit of output. Values β_{ij} are called the coefficients of the direct costs of factors of production, and the matrix B - matrix of direct costs of factors of production composed of these coefficients.

Each column of the matrix $B = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mn} \end{pmatrix}$ defines direct costs of

factors specific to an industry, and every i - line describes the need for system i -factor of production. We suppose that the factors of production are complied the property of linearity and completeness of consumption. If $X = (X_1, X_2, \dots, X_n)$ - the vector of gross output, then total demand of economic system at i -factor

$\sum_{j=1}^n \beta_{ij} \cdot X_j = Z_i$. This relationship can be written in matrix form:

$$Z = B \cdot X = B \cdot S \cdot Y \quad (1.19),$$

$$X = S \cdot Y \text{ and } S = (E - A)^{-1}$$

Matrix $B^* = B \cdot S$ specifies the full costs of production factors per unit of output. As already noted, the amount of each factor is limited and is given by the matrix $D = (d_1, d_2, \dots, d_m)$. The plan to produce the final product is acceptable if the quantity of factors is enough to implement the plan, i.e.:

$$B \cdot S \cdot Y \leq D \quad (1.20)$$

Record balance model with factors of production:

$$\begin{cases} A \cdot X + Y = X \\ X \geq 0 \\ B \cdot S \cdot Y \leq D \end{cases} \quad (1.21)$$

In contrast to the simple balance model this model could be unsolvable even if matrix A is productive matrix, but only if satisfied the relation (1.20). In this case it is impossible to talk about the satisfaction of any final demand.

So before you start solving the system of balance equations it is necessary to check the feasibility of the conditions (1.20) for a given plan Y. If this condition is not met, it is necessary to change the volume of the final product, retaining its structure, i.e. all elements of the plan should be modified in the same number of times. The scaling factor is determined as follows:

$$k = \min_i \left(\frac{d_i}{z_i} \right) \quad i = \overline{1, m} \quad (1.22)$$

Example 2. Use input-output table from example 1 and add 2 factors of production: labour and material resources. Set the table of use of resources:

The other factors	1 division	2 division	3 division	Limit
Labour	10	40	25	60
Materials	50	20	15	200

1. Test the ability to perform new targets Y(250;200;400) under the given limits on the factors of production.
2. Define the volume of final and gross product accessible.
3. Form a plan intersectoral balance table.

Solution.

1. Test the ability to perform new targets Y(250;200;400) under the given limits on the factors of production.

Let`s calculate the matrix of direct costs of factors of production B:

The economic meaning of the elements of the matrix B: b_{ij} - the number of the i- factor necessary to produce a unit of j-product. Each column of the matrix B shows the cost of factors related economic object.

$$B = \begin{pmatrix} \frac{10}{100} & \frac{40}{100} & \frac{25}{100} \\ \frac{50}{100} & \frac{20}{100} & \frac{15}{100} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.4 & 0.125 \\ 0.5 & 0.2 & 0.075 \end{pmatrix}$$

Let`s denote by Z the needs of all objects of the factors of production.

$$Z = B * X = B * S * Y$$

$$\begin{cases} Ax + Y = X \\ X > O \\ BS Y \leq D \end{cases}$$

B^* - matrix of the full costs of factors of production.

$$B^* = \begin{pmatrix} 0.1 & 0.4 & 0.125 \\ 0.5 & 0.4 & 0.075 \end{pmatrix} * \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix} = \begin{pmatrix} 0.3475 & 0.5266 & 0.2736 \\ 0.7316 & 0.3458 & 0.2732 \end{pmatrix}$$

$$Z = B^* * Y = \begin{pmatrix} 0.3475 & 0.5266 & 0.2736 \\ 0.7316 & 0.3458 & 0.2732 \end{pmatrix} * \begin{pmatrix} 250 \\ 200 \\ 400 \end{pmatrix} = \begin{pmatrix} 301.62 \\ 361.33 \end{pmatrix}$$

Under the given constraints on the factors of production scheduled task cannot be performed, since $Z > D$.

2. Define the volume of final and gross product accessible.

Under the given constraints on the factors of production scheduled task cannot be performed, since $Z > D$. You must correct target Y . It is needed to define a new scheduled Y and look for accessible gross output X . Since we assume a condition of completeness consumption not only products, but for factors of production also, let's correct targets Y and X saving the structure of final demand, i.e. the ratio of the components should remain the same.

$$K = \min(D/Z)$$

K - the rate of scale changing.

$$K = \min \left(\begin{pmatrix} 60/301.62 \\ 200/361.33 \end{pmatrix} \right) = \begin{pmatrix} 0.1989 \\ 0.55 \end{pmatrix} = 0.1989$$

$$Y_{cor.} = K * Y$$

$$Y_{cor.} = 0.1989 * (250; 200; 400) = (49.725; 39.78; 79.56)$$

$$X_{cor.} = S * Y_{cor.}$$

$$X_{cor.} = \begin{pmatrix} 1.228 & 0.170 & 0.271 \\ 0.431 & 1.123 & 0.255 \\ 0.418 & 0.483 & 1.156 \end{pmatrix} * \begin{pmatrix} 49.725 \\ 39.78 \\ 79.56 \end{pmatrix} = \begin{pmatrix} 89.4 \\ 86.4 \\ 131.97 \end{pmatrix}$$

3. Let's build a new planned balance:

Division	Intermediate good			Intermediate P	Final good Y	Gross prod. X
	1	2	3			
1	8.94	4.32	26.39	39.65	49.725	89.4
2	26.82	0	19.8	46.6	39.78	86.4
3	17.88	34.56	0	52.44	79.56	132
Итого:	53.64	38.88	46.19		169.1	307.8
V	35.76	47.52	85.81		169.1	
X	89.4	86.4	132			307.8

1.5 Price balance model.

So far our discussion has only touched on the production technology. Consider the balance column and examine the price point of balance models. Let's write down the balance ratios for columns value of intersectoral balance:

$$\sum_{i=1}^n a_{ij} + V_j = X_j \quad j = \overline{1, n} \quad (1.23)$$

Here V_j is the added cost.

Suppose that next year is projected price changes in each industry j in p_j time relative to the current year under the same natural values of the vectors X and Y. The values are called indexes of price changes.

Enter the price index in equation (1.23) substituting $t a_{ij}$ in $a_{ij} = \alpha_{ij} \cdot X_j$. Then (1.23) can be written:

$$\sum_{i=1}^n p_i \alpha_{ij} \cdot X_j + V_j = p_j \cdot X_j \quad (1.23)$$

Divide (1.22) on gross output and receive:

$$\sum_{i=1}^n p_i \alpha_{ij} \cdot + V_j' = p_j, \quad j = \overline{1, n} \quad , (1.24),$$

$V_j' = \frac{V_j}{X_j}$ is the share of added value per unit of j- product.

Price balance model in matrix form can be written:

$$\begin{cases} A^T \cdot P + V' = P \\ P \geq 0 \end{cases} \quad (1.25)$$

Here A^T is the transposed matrix to A matrix of technological coefficients, V' - matrix of added costs per unit of output. In the model are set A^T and V' and calculated a matrix of indexes of price changes P.

If we assume that prices for the products of industries in the reporting period was equal to one, p_j can be interpreted as the unit price of products in the j- industry.

It is easy to establish a relation between the cost model and model output, namely: $X \leftrightarrow P, A \leftrightarrow A^T, Y \leftrightarrow V'$. Referring to these mutual correspondence, the model of output and pricing model is called the dual.

The same theoretical principles is fair for pricing model as for the model of input-output. In particular, if A is productive, there is the only nonnegative solution of the model (1.25):

$$P = (E - A^T)^{-1} \cdot V' \quad (1.26).$$

It can be shown that $(E - A^T)^{-1} = ((E - A)^{-1})^T$, then

$$P = ((E - A)^{-1})^T \cdot V' = S^T \cdot V' \quad (1.27)$$

For a price balance model matrix S^T is a multiplier of the distribution of changes in the share of value added, i.e.

$$\Delta P = S^T \cdot \Delta V' \quad (1.28).$$

Equality (1.28) helps to calculate how value added change will affect the price index in any industry.

In the case where the value added is only represented by wages, price indices are proportional to the coefficients of the total demand of labor regardless of the targets for the final product, and the proportionality coefficient coincides with the coefficient of the wage w_0 , i.e $P = w_0 \cdot b^*$. We will show this.

Let the vector of direct labor cost $b_0 = (b_1^0, b_2^0, \dots, b_n^0)$, then $w_0 \cdot b_j^0$ - wages, when manufacturing units of j- product. Believe that $v_j' = w_0 \cdot b_j^0$, then $P = S^T \cdot V' = S^T \cdot w_0 \cdot b_0 = w_0 \cdot S^T \cdot b_0 = w_0 \cdot b_0 \cdot (S^T)^T = w_0 \cdot b_0 \cdot S = w_0 \cdot b^*$

Therefore, $P = w_0 \cdot b^*$

Example 3.

It is known the following intersectoral balance of the three branch model in natural measurements:

Branch	Intermediate good	Final good

	Agriculture	Manufacturing	Other branches	
Agriculture	10	50	40	200
Manufacturing	120	110	70	140
Other branches	40	80	20	250
Lobour cost	30	50	100	
Profit	200	180	300	

The coefficient of the wage - $\omega_0 = 2$.

Determine:

1. Prices for products of industries, if the added value is included only salary;
2. Prices for products of industries, if the value added in addition to salary include profit;
3. How will the prices change, if the value added in addition to the wage and profit include taxes $N = (5; 10; 20)$.

Solution.

1. Prices for products of industries, if the added value is included only salary.

Branch	Intermediate good			Intermediate P	Final good Y	Gross prod. X
	Agriculture	Manufacturing	Other branches			
Agriculture	10	50	40	100	200	300
Manufacturing	120	110	70	300	140	440
Other branches	40	80	20	140	250	390
Lobour cost	30	50	100			
Profit	200	180	300			

$$\begin{cases} A^T + V' = C \\ C \geq 0 \end{cases} \quad (4)$$

Model (4) is called a model of equilibrium prices.

V' - total value added per unit of product;

P - the prices of the products.

Build the matrix A of technological factors (direct costs).

$$A = \begin{pmatrix} 0.03 & 0.11 & 0.1 \\ 0.27 & 0.25 & 0.18 \\ 0.13 & 0.18 & 0.05 \end{pmatrix} \quad \text{in natural measurements}$$

$$V' = \omega_0 * b_0$$

B_0 is the vector of direct labor costs.

$$B_0 = \left(\frac{30}{300} \quad \frac{50}{440} \quad \frac{100}{390} \right) = (0.57; 0.55; 0.33)$$

$$V' = 2 * (0.57; 0.55; 0.33) = (1.14; 1.1; 0.66)$$

Will build a system of balance equations:

$$\begin{cases} 0.03P_1 + 0.27P_2 + 0.13P_3 + 1.14 = P_1 \\ 0.11P_1 + 0.25P_2 + 0.18P_3 + 1.1 = P_2 \\ 0.1P_1 + 0.18P_2 + 0.05P_3 + 0.66 = P_3 \end{cases} \quad (5)$$

$$\{P_i \geq 0 \quad (6)$$

Constraints (5) and (6) determine the economic-mathematical model of the problem.

Define prices using the Kramer's rule. Simplify the balance equations:

$$\begin{cases} 0.97P_1 - 0.27P_2 - 0.13P_3 = 1.14 \\ -0.11P_1 + 0.75P_2 - 0.18P_3 = 1.1 \\ -0.1P_1 - 0.18P_2 + 0.95P_3 = 0.66 \end{cases}$$

Vector of prices covering only salary: $P(2.2; 2.1; 1.3)$

2. Prices for products of industries, if the value added in addition to salary include profit.

Let's use matrix calculations.

$$P = ((E - A)^{-1})^T \cdot V' = S^T \cdot V'$$

$$S^T = \begin{pmatrix} 1.1 & 0.2 & 0.2 \\ 0.7 & 1.5 & 0.4 \\ 0.3 & 0.3 & 1.1 \end{pmatrix}$$

$$\Pi' = \left(\frac{200}{300} \quad \frac{180}{440} \quad \frac{300}{390} \right) = (0.67; 0.41; 0.77)$$

$$V' = (0.67; 0.41; 0.77) + (1.14; 1.1; 0.66) = (1.8; 1.5; 1.4)$$

$$P = \begin{pmatrix} 1.1 & 0.2 & 0.2 \\ 0.7 & 1.5 & 0.4 \\ 0.3 & 0.3 & 1.1 \end{pmatrix} * \begin{pmatrix} 1.8 \\ 1.5 \\ 1.4 \end{pmatrix} = \begin{pmatrix} 3.5 \\ 3.1 \\ 2.5 \end{pmatrix}$$

3. How will the prices change, if the value added in addition to the wage and profit include taxes $N = (5; 10; 20)$.

$$\Delta P = S^T * \Delta V'$$

$$\Delta V' = \begin{pmatrix} 5 \\ 300 \\ 10 \\ 440 \\ 20 \\ 390 \end{pmatrix} = \begin{pmatrix} 0.017 \\ 0.02 \\ 0.05 \end{pmatrix}$$

$$\Delta P = \begin{pmatrix} 1.1 & 0.2 & 0.2 \\ 0.7 & 1.5 & 0.4 \\ 0.3 & 0.3 & 1.1 \end{pmatrix} * \begin{pmatrix} 0.017 \\ 0.02 \\ 0.05 \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.07 \end{pmatrix}$$

Chapter 2. Theory of Games.

2.1. Basic Terms and Definitions of the Theory of Games

There are several definitions of games, differing by the degree of formality, ranging from simple informative descriptions to the game as a purely abstract mathematical category. But in all cases, game theory considers situations in which the subject does not have all the data necessary for making optimal decisions. These data may be missing due to objective reasons or intentionally hide those who pursues its own goals, i.e. the solution should be taken in conditions of uncertainty. Such situations are called games and the decision-making process in this situation - game.

1. The game is real or formal conflict, which has at least two members, each of which aims to achieve its own goals. Moreover, all the participants can't achieve their goals to the maximum extent. Thus, the actions of some players create for others the situation of uncertainty and the task of each participant is possible success in such an uncertain situation.

The concept of "maximum success" needs to be specifically defined in each specific case. This can be the maximum profit, minimum cost, etc. Such expressions of success, prestige, authority, influence, non-quantitative in game theory are not considered.

2. In a broad sense, the game can be called any conflict, any conflict of interest, as well as any situation in which you have to overcome obstacles, which, generally speaking, cannot be remedied, but the impact of which can be reduced by choosing a suitable course of action.

From the real conflict game differs. It is conducted according to certain rules. These rules determine:

- participants of the game,
- the rights and obligations of participants (possible actions),
- the outcome of the game (win or loss of each participant, depending on the situation).

People have long used such formalized models of conflict - games in the literal sense of the word: chess, checkers, card games. It is source of the name "game theory" and its terminology.

Let's introduce the basic concepts of game theory:

1. A player is considered as one participant or group of participants, if they all have a common goal, which does not coincide with the goals of other groups. The player may be the environment (nature), serving as a set of disruptive circumstances. For example: football has 2 participants (team acts as a single player).

2. Rules or the rules of the game determine the possible lines of conduct players, elections and moves for players at any stage of game development. To make the choice of the player is to stay at one of the possible lines of conduct. Selects the player with moves.

A set of rules that determines the choice of action in each individual course to the player depending on the situation, is called a strategy of the player.

Strategy is a set of directions to any state of information available to the player at any stage of game development. Strategies can be good and bad, successful and unsuccessful. The game may have one, two, many strategies.

For example, the game of chess has many strategies. The strategy in the game of chess should indicate to the player what move he should do in any game development. When the game has many strategies usually it is impossible to go through them, therefore, for analysis and the study of the strategies chosen by the principal. For different players main strategy are of different strategies, as a rule. In the game, reflecting the economic situation, the strategies can be the size of capital investments, the choice of a certain value stocks, certain pricing etc.

The object of game theory is to identify the best, the “optimal” strategies. The strategy is called optimal if it provides the player with the best position in the game, i.e. the maximum benefit regardless of the actions of other players.

3. Winning is a measure of effect for the player. In the games, reflecting the economic situation, the gain is usually measured in terms of value (profit, profitability, cost, etc.) .The card games effect is often measured in monetary terms, sports glasses, etc.

Again note that game theory considers only those situations where winning is the quantitative expression.

2.2. Matrix Games.

A matrix game is called the game of two players with a zero amount.

Matrix game can be considered as the following abstract game for two players.

The first player has m strategies ($i=1, 2, \dots, m$), the second has n strategies $j=(1, 2, \dots, n)$. Each pair of strategies (i, j) is set in accordance with the number a_{ij} expressing the first to win for the second player if the first player will use its i -strategy, and the second is its j -strategy.

Each player makes one move: the first player chooses the i -strategy ($i=1, 2, \dots, m$), the second is its j -th strategy $j=(1, 2, \dots, n)$, then the first player receives a payout a_{ij} for the account of the second player (if $a_{ij} < 0$, it means that the first player pays the second sum $|a_{ij}|$). The game is over.

Each strategy of player ($i=1, 2, \dots, m$), $j=(1, 2, \dots, n)$ is often called a pure strategy.

Obviously the matrix game belongs to antagonistic games, as the first win is equal to the loss of the second player. From its definition it follows that for formalizing matrix game it is enough to determine the matrix $A=(a_{ij})$ of order $m \times n$ of wins the first player. If we consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix},$$

the conduct of each party of the matrix game with matrix A is reduced to the choice by the first player of the i -row, and the second player of the j -column and get the first player (due to the second) win a_{ij} , locating in the matrix A at the intersection of the i -row and j -column.

To formalize the real conflict in the form of a matrix game it is need to emphasize and renumber the pure strategy of each player and make a matrix of the winnings.

The next step is to identify the best strategies and winnings.

Consider first a simple example of formalization of conflict situations.

Example 4.

The company commissioned to produce two types of perishable products P1 and P2. Daily expenses on the production and sale of products shall not exceed 4000 RUR. Management task is to determine the daily production volume of each product, with the aim to obtain the highest profit. This study was conducted, which showed the following:

- cost per unit of product P1 is equal to 0.8 RUR, sale price -1,2 RUR; the unit cost of product P2 is equal to 0.5 RUR, sale price - 0.8 RUR;
- if the products are not sold at the day of release, its quality is greatly reduced, and it will sell next day for the price in 4 times less;
- sales of products depends on weather condition - in good weather, is implemented 1000 units of product P1 and 6,000 units P2 ; in bad weather is being implemented 4,000 units of product P1 and 1200 units of P2;
- implementation of all the produced products consumed 200 RUR

So, it is important for the enterprise to know the weather condition. Then it should produce products in such amount and range that it would realize the most at the same day. If it were possible to predict in advance the weather conditions, then the optimal production plan would be a plan that is fully focused on the well-known weather condition. However, there are currently no reliable methods of weather forecast, and the company must plan under uncertainty. You can interpret the situation as follows: on the one hand the company is interested to produce products with the greatest benefit for themselves, on the other hand, the enemy is nature, which can maximize the damage to the enterprise. Therefore, this situation can be considered as the situation for the antagonistic game of two players: the first player is the company's, the second - nature.

You can, of course, to assume that nature is not a reasonable opponent and it will not be to study the behavior of the enterprise in order to maximize the damage to him and therefore should not be considered in this situation as a zero-sum game. Such arguments have reason, then you can learn statistics about the behavior of weather and build a plan of production taking into account the weather on average.

However, such approach has its advantages. Indeed, considering the nature as the enemy, the company could build its optimal plans taking into account the most adverse action of nature, and if nature departs from these their most unfavorable for the company's actions, then this will be the optimal plan to conduct the enterprise and will enable them to increase their profit.

So, in this situation there are two players: man and nature. What are their strategies? It is obvious from the nature there are two strategies: the first – to create good weather, the second - to create bad weather. The company also has two strategies: the first - to produce products based on good weather, the second - to produce waiting for the bad weather.

Thus, both players have two strategies (a finite number of strategies), so we come to the game of two players with zero-sum, i.e. the matrix game with matrix A

weather ↙	good weather	bad weather
enterprise		
good weather	a_{11}	a_{12}
bad weather	a_{21}	a_{22}

Elements a_{ij} ($i=1, 2; j = 1, 2$) in the matrix A express the profit of the company provided if the company applies its i-strategy, and the nature - its j-strategy.

Make calculations of the elements a_{ij} ($i=1, 2; j = 1, 2$).

The profit P is $P=Z-C$,

Z is the amount of revenue through the sales of products, C is production and sales costs.

Calculations will be made for a period of one day. To obtain the element a_{11} is necessary to consider that the company applies its first strategy, i.e. takes the calculation of the nice weather and produces 1000 units of product P1 and 6000 units of product P2, therefore, the costs of C1 will be:

$$C1 = 1000 * 0,8 + 6000 * 0,5 + 200 = 4000 \text{ RUR}$$

The nature also applies its first strategy, i.e. the weather is good, then the company on the same day implements all the products on the selling price and will receive an amount

$$Z1 = 1000 * 1,2 + 6000 * 0,8 = 6000 \text{ RUR}$$

Thus, in this case the profit of the company will be

$$a_{11} = Z1 - C1 = 6000 - 4000 = 2000 \text{ rubles}$$

To obtain a_{12} note that the company takes into count a good weather, i.e., applies its first strategy, and the nature applies its second strategy, i.e. the weather is bad. In this case, the cost will be the same, i.e., $C1 = 4000$ rubles, and the amount of revenue Z will be different. It should be noted that during bad weather in the same day is implemented 4000 units of product P1 and produced only 1000 units, i.e. all products P1 will be sold at a price of 1.2 rubles. For product P2 is the following: if the weather is bad it is implemented in the same day 1200 units at a price of 0.8 RUR and other $6000 - 1200 = 4800$ units are implemented on the next day at a price of $0,8 : 4 = 0,2$ per. Thus, the amount of revenue Z in this case will be

$$Z_2 = 1000 \cdot 1,2 + 1200 \cdot 0,8 + 4800 \cdot 0,2 = 3120 \text{ RUB}$$

Thus, in this case the profit of the company will be

$$a_{12} = 3120 - 4000 = -880 \text{ rubles,}$$

i.e. in this case the company will incur a loss 880 rubles

Now the company will use its second strategy, i.e., takes into account the bad weather, then there will be produced 4000 units of product P1 and 1200 units P2 and its costs will be:

$$C_2 = 4000 \cdot 0,8 + 1200 \cdot 0,5 + 200 = 4000 \text{ RUR}$$

If the weather is bad, i.e., nature will have their second strategy, all manufactured products will be sold on the same day, and the company will have revenue

$$Z_3 = 4000 \cdot 1,2 + 1200 \cdot 0,8 = 5760 \text{ RUR his profit will be}$$

$$a_{22} = Z_3 - C_2 = 5760 - 4000 = 1760 \text{ RUR}$$

If an enterprise applies its second strategy, and the nature of their first strategy, i.e., will be produced 4000 units P1 and 1200 units of P2, and on the same day will be sold 1000 units P1 1.2 RUR and 1200 units of P2 at a price of 0.8 rubles, and on another day 4000 - 1000 = 3000 units P1 and costs 0.3 rubles. In this case, the amount of revenue will be:

$$Z_4 = 1000 \cdot 1,2 + 1200 \cdot 0,8 + 3000 \cdot 0,3 = 3060 \text{ RUR}$$

and the profit $P = Z_4 - C_2$, i.e., $A_{21} = 3060 - 4000 = -940$ rubles

Thus, the matrix A takes the form:

$$A = \begin{pmatrix} 2000 & -880 \\ -940 & 1760 \end{pmatrix}$$

The solution of this game will consider in the future.

2.3. Solution of Matrix Games in Pure Strategies.

Notion of optimal strategies of the players is important in the study of games. This concept intuitively embedded in this sense: the strategy of the player is optimal if the application of this strategy provides the most guaranteed win at all possible strategies of the other player.

Based on these positions, the first player explores the matrix A of his winnings as follows: for each pure strategy of each value i (i = 1, 2, ..., m) is determined by the minimum value of the gain depending on the strategies of the second player:

$$\min_j(a_{ij}), (i = 1, 2, \dots, t),$$

i.e. is determined by the minimum prize for the first player, provided that he will use his i -th pure strategy, then those minimal winnings found this strategy $i = i_0$ at which this minimum payout will be maximum, i.e., it is

$$\max_i(\min_j a_{ij}) = a_{i_0 j_0} = \alpha \quad (2.1)$$

The number is defined by the formula (2.1) is called the lower net price of the game. It shows what is the minimum payout can guarantee yourself the first player using their pure strategies for all the actions of the second player.

The second player at his optimal behavior should strive as possible at the expense of its strategies to minimize the first to win. So for the second player is found ,

$$\max_j(a_{ij})$$

i.e. is determined by the maximum jackpot of the first player, provided that the second player will use its j -th pure strategy, then the second player finds this j -th strategy in which the first player will receive a minimum gain, i.e., finds

$$\min_j(\max_i(a_{ij})) = a_{i_0 j_0} = \beta \quad (2.2)$$

The number is defined by the formula (2.2) is called a pure top price of the game and shows what the maximum gain at the expense of their strategies can guarantee for the first player.

In other words, the first player can win not less than α by applying his pure strategy, and the second player may not allow the first to win more than β through the use of their pure strategies. You can prove that is always performed inequality:

$$\max_i(\min_j a_{ij}) \leq \min_j(\max_i(a_{ij})) \quad (2.3)$$

If in the game with matrix A , the lower and upper net prices of the game are the same, i.e. it is true

$$\alpha = \max_i(\min_j a_{ij}) = \min_j(\max_i(a_{ij})) = \beta \quad (2.4)$$

they say that the game has an optimal solution in pure strategies, while the number of $V = \alpha = \beta$ is what is called the net price of the game, and the item is called a saddle point for which the relation (2.4) is correct.

Thus, the saddle point defines a pair of pure strategies (i_0, j_0) respectively of the first and second players that equality $\alpha = \beta$ has been attained. These strategies are called optimal pure strategies.

The concept of saddle point invested the following sense: if one player's strategy corresponding saddle point, then the other player will not be able to do better than stick to the strategy corresponding to the saddle point, otherwise it will

get worst result. These conditions can be written mathematically in the following proportions:

$$a_{i_0} \leq a_{i_0 j_0} \leq a_{i_0 j} \quad i = \overline{1, m}; \quad j = \overline{1, n} \quad (2.4)$$

Thus, from (2.4), the saddle element is minimal in the line i_0 and the maximum in the column j_0 of a matrix A .

The determination of the saddle point of a matrix A is as follows: in each row of the matrix A sequentially find the minimum element and check whether this element is maximum in its column. If he it is so, it is the saddle element and the pair of the corresponding strategies determines the optimal solution of the game in pure strategies.

Pair of pure strategies (i_0, j_0) of the first and second players, corresponding with saddle element, and the number of $V = a_{i_0 j_0}$ is called the optimal solution of the matrix game in pure strategies.

Example 5.

The game is defined by the following matrix

$$A = \begin{pmatrix} 5 & 3 & 4 & 3 \\ 7 & 2 & 0 & -2 \\ 10 & -1 & -4 & 2 \end{pmatrix}$$

Find the solution of this game.

Solution.

Consider the first line, and defined it minimal elements. This element is 3 in the second and fourth columns. Check whether they are maximum in the second and fourth columns. It turns out so. Therefore, they are saddle elements and form a solution of the game: the value of the game $V = 3$, the optimal pure strategy of the first player is the first. Optimal pure strategies of the second player are the two - second and fourth.

Further, in the second line, the minimum element is 2 and is not maximal in its fourth column, means that it cannot be a saddle. In the third line, the minimum element is -4 and is not maximal in the third column, therefore, it is not the saddle.

Thus, consistently analyzed all the rows of the matrix of wins, and it turned out there is no more saddle points in this game. Net lower price the game is $\alpha = 3$, clean the top price of the game is $\beta = 3$. They are the same.

Not every matrix game has a payoff matrix containing the saddle point, hence, not every matrix game has an optimal solution in pure strategies.

Let us consider the following example.

Example 5.

Game coins.

Payment matrix game with coins is A.

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Find the solution of this game.

Solution.

The matrix A has no saddle point. Lower the price of games is $\alpha = -1$; upper value of the game is $\beta = 1$. Therefore the game has no optimal solution in pure strategies, because $\alpha \neq \beta$.

2.4. The Concept of Mixed Strategy

The examples in the previous paragraph show that not every matrix game has an optimal pure strategy. The study of the matrix game starts with finding her saddle point. If a matrix game has a saddle point, it is end point in the study of games. If a matrix game has no saddle point, it is not the optimal solution in pure strategies. In this case, you can find the lower and upper net price of this game, which indicate that the first player should not hope to win more than the top price of the game α , and can be sure in getting the win is not lower than the price of the game β .

Improving decision matrix games should be sought in the use secrecy of use pure strategies and the possibility of repeated games as parties. So, for example, there is a series of games of chess, checkers, football, and every time the players apply their strategies in such a way that their opponents are not even aware of their content, and in this way the average reach certain winnings by playing the whole series of parties. These wins will be at the average more than the bottom price of the game and less than the high price of the game. More than this average value, the better the strategy applies a player. So the idea arose to use pure strategies randomly with a certain probability. This assures the secrecy of their application. Each player can change the probability of use of their pure strategies so as to maximize his average gain and obtain the optimal strategy in this way. This idea led to the concept of a mixed strategy.

A mixed strategy for a player is called a complete set of probabilities of applying his pure strategies.

Thus, if the first player has m of pure strategies ($i = 1, 2, \dots, m$), his mixed

strategy is a vector $\bar{X} = (x_1, x_2, \dots, x_m)$ whose coordinates satisfy the following relationships: $x_i \geq 0, i = \overline{1, m}$ and $\sum_i x_i = 1$

Similarly for the second player, which has n pure strategies ($j = 1, 2, \dots, n$), mixed strategy is a vector $\bar{Y} = (y_1, y_2, \dots, y_n)$ whose coordinates satisfy the following relationships:

$$y_j \geq 0, j = \overline{1, n} \quad \text{and} \quad \sum_j y_j = 1.$$

Pure strategies are incompatible events, because the application of one pure strategy eliminates the use of another every time. In addition, they are the only possible events.

Obviously, a pure strategy is a special case of a mixed strategy. Indeed, if in a mixed strategy any k -th pure strategy is applied with probability one, all other pure strategies do not apply (apply with probability 0) and this k -th pure strategy is a special case of a mixed strategy.

The average gain (win) of the first player in a matrix game with matrix A is defined as the mathematical expectation of winning this player:

$$E(X, Y) = X \cdot A \cdot Y^T = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_i y_j \quad (3.1)$$

Obviously, the average gain of the first player is a function of two sets of variables \bar{X} and \bar{Y} . The first player has a goal to maximize their average gain \bar{X} due to a change in their mixed strategies, and the second is committed to make minimum \bar{Y} due to their mixed strategies

2.5. The Solution of the Matrix Game in Mixed Strategies.

The decision matrix games without saddle point can be found, if you build the payoff function, and then solve the problem of finding its maximum or minimum. But when a large number of strategies this way is very time consuming and almost never used.

A versatile method for the solution of matrix games is the solution of a game using a linear programming method.

The Creator of game theory J. von Neumann stated that each matrix game can be represented as a linear programming problem and vice versa.

Let the matrix game given by the matrix A :

$$A \begin{pmatrix} 4 & 3 & 4 & 2 \\ 3 & 4 & 6 & 5 \\ 2 & 5 & 1 & 3 \end{pmatrix}$$

Solution.

The matrix A has no saddle point, so looking for a solution in mixed strategies by reduction to linear programming problem. We introduce mixed strategy:

$$X^* = (x_1^*, x_2^*, \dots, x_m^*) \text{ and } Y^* = (y_1^*, y_2^*, \dots, y_n^*)$$

We must determine the optimal strategy of the first player and form the following problem linear programming:

$$\text{find min } Z = x_1 + x_2 + x_3$$

$$\text{with limitations } \begin{cases} 4x_1 + 3x_2 + 2x_3 \geq 1 \\ 3x_1 + 4x_2 + 5x_3 \geq 1 \\ 4x_1 + 6x_2 + x_3 \geq 1 \\ 2x_1 + 5x_2 + 3x_3 \geq 1 \end{cases}$$

$$x_i \geq 0, \quad i = \overline{1,3}, \quad x_i = \frac{x_i^*}{V}$$

Dual task is formulated as follows to determine the optimal strategy of the second player:

$$\text{find max } W = y_1 + y_2 + y_3 + y_4$$

$$\text{with limitations } \begin{cases} 4y_1 + 3y_2 + 4y_3 + 2y_4 \leq 1 \\ 3y_1 + 4y_2 + 6y_3 + 5y_4 \leq 1 \\ 2y_1 + 5y_2 + y_3 + 3y_4 \leq 1 \end{cases}$$

$$y_j \geq 0, \quad j = \overline{1,4}, \quad \text{где } y_j = \frac{y_j^*}{V} \quad (4.13)$$

Let's solve the problem using the simplex method. Simplex table will make up for the task 2, so it limits the type is \leq . Additional variables entered in the constraints of this task will form the basis of the first simplex table.

Table 2.

i	Basis	B	Y ₁ .	Y ₂	Y ₃	Y ₄	Y ₅	Y ₆	Y ₇
1	Y ₅	1	4	3	4	2	1	0	0
2	Y ₆	1	3	4	6	5	0	1	0
3	Y ₇	1	2	5	1	3	0	0	1
M+1	Δ	-1	-1	-1	-1	0	0	0	

1	Y ₁	1/4	1	3/4	1	1/2	1/4	0	0
2	Y ₆	1/4	0	7/4	3	7/2	-3/4	1	0
3	Y ₇	1/2	0	7/2	-1	2	-1/2	0	1
M+1	Δ	1/4	0	-1/40	0	-1/2	-1/4	0	0

.....

1	y ₁	3/14	1	1/2	4/7	0	5/14	-1/7	0
2	y ₄	1/14	0	1/2	6/7	1	-3/14	2/7	0
3	y ₇	5/14	0	5/2	-19/7	0	-1/14	-4/7	'
m+1	Δ	2/7	0	0	3/7	0	1/7	1/7	0

The optimal plan task 2 has the form: $Y=(3/14,0,0,1/4)$;

$$W_{\max} = \frac{1}{V} = \frac{2}{7} \rightarrow V = \frac{7}{2}.$$

The optimal solution of problem 1, the dual to problem 2 is from a string of recent estimates simplex table:

$$X=(1/7,1/7,0); Z_{\min} = \frac{1}{V} = \frac{2}{7} \rightarrow V = \frac{7}{2}.$$

Now you need to make a change of variables in accordance with the formulas (4.12) and (4.13).

$X^*=(1/2,1/2,0)$; $Y^*=(3/4,0,0,1/4)$. The value of the game $V=7/2$

Exam questions

1. The concepts: model and modelling.
 2. Classification models.
 3. The use of modeling in economic research.
 4. Peculiarities of mathematical modeling of economic processes.
 5. General view of mathematical models and main directions of their research.
 6. Methods of multi-criteria optimization.
 7. Simulation experiments and simulation system.
 8. The production function (General concepts, definitions, types of production functions). A variety of production capacities.
 9. The basic properties of production functions. The concept of the economic field.
 10. The concept of expanding the scale of production. Changes in output and their relationship with the elasticity of production.
 11. The concept of resources substitution. Marginal rates of substitution of resources. Elasticity of substitution resources.
 12. The concept of the balance of the manufacturer. The method of Lagrange for solving the conditional extremum.
 13. Determining the optimal production in conditions of limited resources.
 14. The main types of production functions of the issue. (A linear function of the Cobb-Douglas production function, the function Leontiev)
 15. A simple model of the economy.
 16. The concept of demand. The main approaches to the description of demand.
 18. The concept of the utility function. The basic properties of the utility function.
 19. The model of consumer behavior.
 20. Curves (the surface) of indifference (the concept, basic properties).
 21. The concept of replacement goods.
 22. The analysis of the model of consumer behavior using Lagrange function.
- Optimal choice.
23. Demand function and demand functions based on the model of consumer behavior.
 24. The interindustry balance.
 25. Static balance model of production. Basic assumptions underlying the build balance model.
 26. The system of the balance equations. The productivity of the technological matrix coefficients (definition, criteria of productivity, properties productive matrix).
 27. The matrix of the full costs (economic sense, methods of calculation). Matrix indirect costs (definition, methods of calculations, the economic sense).

28. The factors of production. Balance model with production factors.
29. The model of equilibrium prices.
30. The concept of equilibrium prices. Theorem on equilibrium prices.
31. The notion of economic equilibrium. Static and dynamic balance.
32. The balance in different markets. The notion of General economic equilibrium.
35. Simple model of the economy.
36. Two-sector model of economic equilibrium
37. Spider model of market equilibrium.
38. The concept and factors of economic growth.
39. Model of economic growth without technical progress (Domar, Harrod's)
40. The neoclassical model of economic growth without technical progress Solow.
41. The concept of technological progress and the form of its expression.
42. Methods of description of technical progress on Hicks, Harrod's and Solow.
43. Conditions of equilibrium growth with technical progress.
44. General formulation of the problem of dynamic programming. The principle of optimality of Belman and its implementation.

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Марина Владимировна **Кемаева**

МАТЕМАТИЧЕСКИЕ МОДЕЛИ В ЭКОНОМИКЕ

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университет им. Н.И. Лобачевского».
603950, Нижний Новгород, пр. Гагарина, 23.